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EXPONENTIAL STABILIZATION OF A TRANSVERSELY VIBRATING BEAM VIA BOUNDARY CONTROL

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This paper presents a solution to the boundary stabilization of a beam in free transverse vibration. The dynamics of the beam are presented by a non-linear partial differential equation (PDE). A linear control law is constructed to stabilize the beam. The control force consists of feedback from the slope and velocity at the boundary of the beam. The novelty of this article is that it has been possible to stabilize exponentially a free transversely vibrating beam via boundary control without resorting to truncation of the model

1. INTRODUCTION

This article describes how a vibrating beam can be made exponentially stable by using boundary control. The article is limited to free vibrating beams. The vibration occurs in the transverse direction of the beam; hence, the bending stiffness of the beam must be included in the discussion. The novelty of this article is that it is possible to stabilize exponentially a transversely vibrating beam by using boundary control. That is, all control input is applied at one end of the beam. The required measurements are the slope and velocity at the boundary of the beam.

This research is motivated by the industrial interest in active control of vibrating slender bodies. Examples of practical applications where tensioned beams are exposed to undesirable transverse vibrations are: pretensioned marine risers used in off-shore oil and gas exploration, free hanging underwater pipelines, and drill strings for oil and gas exploration.

Boundary control is an efficient method to exclude the effect of both observation and control spillover [1, 2]. A brief review of boundary control is given in reference [3]. Boundary control of flexible systems has been studied by several researchers. In references [4–6], it is shown that feedback from the velocity at the boundary of a string can stabilize the vibration in the string. In reference [7], the asymptotic and exponential stability of an axially moving string is proved by using linear and non-linear state feedback boundary control respectively. It is proved that, in non-linear feedback case, the mechanical energy of the system decreases exponentially. In reference [3], a boundary feedback state is used to control the vibration of an axially moving string. The feedback state includes only the displacement, velocity and slope on the right-hand side of the string. In both [7, 3]

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the control laws are implemented via a mass–damper–spring on the right-hand side of the string. In reference [8], a control strategy, called direct strain feedback, is used to control the vibration of a flexible arm which is modelled as a beam. This control law introduces damping into the governing equation and thus attenuates the vibration. The semigroup and operator theory are used to prove the stability of the system. In reference [9], a control law consisting of feedback from shear force at the root end of an elastic arm is used to control the vibration of the arm. Exponential stability of the closed-loop system is shown.

The novelty of this article is that it has been possible to stabilize a free transversely vibrating beam exponentially via boundary control without resorting to truncation of model. Exponential stability has been proved using a Lyapunov functional.

This paper is arranged as follows: in Section 2 the dynamics of a vibrating beam is presented. Section 3 is devoted to mathematical preliminaries. The boundary control law is derived and exponential stability of the beam is proved in section 4. This paper concludes with some remarks regarding implementation.

2. EQUATIONS OF MOTIONS

The dynamic equation of motion of a modified, non-linear Euler–Bernoulli beam with axial tension $P(x, t)$ and transverse force density $f(x, t)$ can be written as follows:

$$\rho A\eta_{tt}(x, t) + \frac{\partial^2}{\partial x^2}(EI\eta_{xx}(x, t)) - \frac{\partial}{\partial x}(P(x, t)\eta_x(x, t)) - f(x, t) = 0 \tag{1}$$

$\forall(x, t) \in (0, L) \times [0, \infty)$, $\eta(x, t)$ represents the transverse displacement, $f(x, t)$ is the external transverse force distribution on the beam, EI is the beam's stiffness or flexural rigidity and ρA is the weight per unit length. Both EI and ρA are assumed to be constant throughout this article. Note that the axial tension $P(x, t)$ is a function of both time and space. This occurs frequently in practical situations; for instance, pretensioned marine risers exposed to axial wave and current loads. The discussion will be limited to freely vibrating beams; hence $f(x, t)$ is set to zero. In equation (1), notations $\eta_{tt}(x, t) = \partial^2\eta(x, t)/\partial t^2$, $\eta_{xx}(x, t) = \partial^2\eta(x, t)/\partial x^2$ and $\eta_x(x, t) = \partial\eta(x, t)/\partial x$ are used. A beam with its dynamic and geometric boundary conditions is shown in Figure 1.

The axial strain–displacement relationship is given by

$$P(x, t) = P_0 + \tfrac{1}{2}EA\eta_x^2(x, t), \tag{2}$$

where E is the Young modulus and A is the cross-sectional area of the beam. P_0 is the constant axial pretension at the boundary $x = L$. Equation (2) has been used in the literature to describe the variation of tension along the length of a string [6, 10]. In this paper, only the elongation of the beam due to bending is considered. The variation of the length of the beam due to axial force is assumed to be small and negligible. Substitution of equation (2) into equation (1), setting $f(x, t) = 0$ and carrying out the differentiation gives

$$\rho A\eta_{tt}(x, t) + EI\eta_{xxxx}(x, t) - P_0\eta_{xx}(x, t) - \tfrac{3}{2}EA\eta_x^2(x, t)\eta_{xx}(x, t) = 0 \tag{3}$$

which describes the free vibration of the beam subjected to axial tension. The boundary conditions are

$$EI\eta_{xx}(0, t) = EI\eta_{xx}(L, t) = 0, \quad \eta(0, t) = 0, \tag{4a, b}$$

$$u(t) = -EI\eta_{xxx}(L, t) + P_0\eta_x(L, t) + \tfrac{1}{2}EA\eta_x^3(L, t), \tag{4c}$$

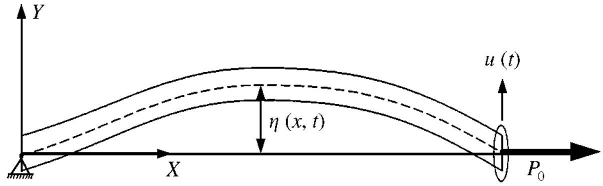

Figure 1. A beam in bending vibration with axial tension.

where the boundary condition (4a) represents the bending moments at the boundaries. Equation (4c) denotes the shear force at $x = L$ of the beam and $u(t)$ represents the boundary control force applied at $x = L$. The boundary condition (4c) represents the balance of the shear force and the control force $u(t)$. The initial conditions are

$$\eta(x, 0) = g_1(x), \quad \eta_t(x, 0) = g_2(x) \tag{5, 6}$$

for all $(x, t) \in (0, L) \times [0, \infty)$. Equations (5, 6) denote the initial position and velocity functions respectively.

The main goal of this paper is to construct a control law, $u(t)$, which stabilizes the non-linear equation of the beam equations (3) and (4a–c) and guarantees that $\eta(x, t) \to 0$ exponentially as $t \to \infty$ for all $x \in [0, L]$.

The following assumption will be made:

(1) $P_0 > 0$ for all $t \geqslant 0$. This assumption simply means that an axially tensioned beam is considered. When $P_0 < 0$ a compressed beam will be considered, but this is not within the scope of this paper.

3. PRELIMINARIES

In order to apply Lyapunov's stability theorem to distributed parameter systems it is necessary to introduce some definitions. Furthermore, a theorem on which the stability proof is based will be presented.

Consider a dynamic system whose state at any fixed time t is specified by $\mathbf{q}(t)$, an element of a state space Γ on which a metric ρ is defined. The distance between two arbitrary states $\mathbf{q}$ and $\mathbf{q}'$ in Γ at time t is specified by $\rho(\mathbf{q}(t), \mathbf{q}'(t))$. These two states are regarded as identical when $\rho(\mathbf{q}(t), \mathbf{q}'(t)) = 0$. For a distributed parameter dynamic system defined on a spatial domain Ω, $\mathbf{q}$ corresponds to a set of real-valued function $\{u_i(x, t)\}$, $i = 1, \ldots, N$ defined in Ω, or an element of a function space $\Gamma(\Omega)$, where x is the spatial coordinate vector (defined in Ω).

Definition 1. An equilibrium state $\mathbf{q}_{eq}$ of a dynamic system is an element of the state-space Γ such that $\rho(\phi(t, t_0)\mathbf{q}_{eq}, \mathbf{q}_{eq}) = 0$ for all $t \geqslant 0$ (the distance of its corresponding trajectory to that state is zero), where $\phi(t, t_0)$ is a continuous operator on Γ, and for any fixed $[t, t_0]$ it maps Γ into itself. The set of all equilibrium states will be called the equilibrium set.

Definition 2. An invariant set $\mathscr{M}$ of a dynamic system is a subset of Γ so that for any initial state $\mathbf{q}(t_0) \in \mathscr{M}$, its corresponding trajectory will remain in Γ for all $t \geqslant t_0$.

Definition 3. An asymptotically invariant set, $\mathscr{M}$, of a distributed parameter dynamic system is uniformly asymptotically stable if

$$\rho(\phi(t, t_0)\mathbf{q}(t_0), \mathscr{M} \to 0 \quad \text{as } t - t_0 \to +\infty$$

uniformly with respect to $t_0 \geqslant 0$ when $\rho(\mathbf{q}(t_0), \mathscr{M} < \delta_2$, where $\delta_2 > 0$ is sufficiently small and $\phi(t, t_0)\mathbf{q}(t_0)$ is the solution of the dynamic system at time t, starting at t_0.

Definition 4. A stable invariant set, $\mathscr{M}$, of a distributed parameter dynamic system is exponentially stable if $\rho(\phi(t, t_0)\mathbf{q}(t_0), \mathscr{M})$ tends exponentially to zero for all $t \geqslant 0$.

The following theorem is taken from reference [11] and readers are referred to this reference for proof of the theorem. A similar theorem for asymptotic stability of distributed parameter system is given in reference [12].

Theorem 1 (Zubov [11]). *In order for an invariant set $\mathscr{M}$ a dynamic system to be stable it is necessary and sufficient that there exist a one-parameter family of functions $V(t)$, having the following properties*:

1. *On any element $\mathbf{q} \in S$ there is defined a function $V(\mathbf{q}, t)$ of the real argument t, defined for $t \geqslant t_0$, where*

$$S = \{\mathbf{q} \in \Gamma | 0 < \rho(\mathbf{q}, \mathscr{M}) < r\}.$$

2. *For any sufficiently small $c_1 > 0$ it is possible to find a quantity $c_2 > 0$ such that $V(\mathbf{q}, t) > c_2$ for $\rho(\mathbf{q}(t_0), \mathscr{M}) > c_1$ and all $t \geqslant 0$.*
3. *$V(\mathbf{q}, t) \to 0$ uniformly relative to $t \geqslant 0$ as $\rho(\mathbf{q}, \mathscr{M}) \to 0$.*
4. *The functional $V(t)$ evaluated along the solution of the system does not increase for all $t \geqslant t_0$ for which it is defined, $\dot{V}(t) \leqslant 0$.*
5. *Furthermore, if the functional $V(t)$ evaluated along the solution of dynamic system tends to zero at $t \to +\infty$ for all $t_0 \geqslant 0$ and $\rho(\mathbf{q}, \mathscr{M}) < \delta_1$, where $\delta_1 > 0$ is sufficiently small, then the invariant set of the dynamic system will be asymptotically stable, and, conversely, if the invariant set is asymptotically stable, this holds, $\dot{V}(t) \leqslant 0$.*

Note that Item 2 in the above theorem indicates positive definiteness of the function $V(\mathbf{q}, t)$. Item 3 requires that the function $V(\mathbf{q}, t)$ admit an infinitesimally upper limit. To prove stability of an underlying distributed parameter system one has to show that there exists a functional with the following properties:

1. the functional is positive definite with respect to a specified metric;
2. the functional admits an infinitesimally upper limit; and
3. the time derivative of the functional along the solutions of the underlying system is negative definite.

4. DESIGN OF BOUNDARY CONTROL LAW

The control objective is to design a control law which guarantees the stability of a continuous system consisting of equations (3) and (4a–c), using the boundary control equation (4c). Assumption 1 will be used throughout the paper to establish the stability property of the system when designing the control law. The following Lyapunov function is introduced:

$$V(t) = \frac{\rho A}{2}\int_0^L \eta_t^2(x,t)\,\mathrm{d}x + \frac{P_0}{2}\int_0^L \eta_x^2(x,t)\,\mathrm{d}x + \frac{EA}{8}\int_0^L \eta_x^4(x,t)\,\mathrm{d}x + \frac{EI}{2}\int_0^L \eta_{xx}^2(x,t)\,\mathrm{d}x + \gamma\rho A\int_0^L x\eta_t(x,t)\eta_x(x,t)\,\mathrm{d}x \tag{7}$$

$\forall t \geqslant 0$, where γ is a small positive constant real number and $\eta(\cdot,\cdot)$ satisfies the boundary-value problem, equations (3) and (4a–c). A metric is defined as

$$\rho(\mathbf{q},\mathbf{0}) = \left[\frac{\rho A}{2}\int_0^L \eta_t^2(x,t)\,\mathrm{d}x + \frac{P_0}{2}\int_0^L \eta_x^2(x,t)\,\mathrm{d}x + \frac{EA}{8}\int_0^L \eta_x^4(x,t)\,\mathrm{d}x + \frac{EI}{2}\int_0^L \eta_{xx}^2(x,t)\,\mathrm{d}x\right]^{1/2}, \tag{8}$$

where $\mathbf{q}^{\mathrm{T}} = [\eta_t\,\eta_x\,\eta_{xx}]$. The metric $\rho(\mathbf{q},\mathbf{0})$ establishes a measure of the closeness of the state $\mathbf{q}$ to the equilibrium null state in terms of the velocity, curvature and slope of the beam. In addition, the metric ρ corresponds to a measure of the total energy of the system; more specifically, $\rho(\mathbf{q},\mathbf{0})$ corresponds to $\sqrt{2E(t)}$, where $E(t)$ is the total energy of the system. The first term represents the kinetic energy, the second and third terms represent the potential energy due to the axial force and the last term represents the potential energy due to bending.

Theorem 2. *Let the boundary control law* $u(t)$, *equation* (4c), *be*

$$u(t) = -K_1\eta_x(L,t) - K_2\eta_t(L,t), \tag{9}$$

where the feedback gains K_1 *and* K_2 *are selected according to*

$$K_1 \geqslant -\frac{1}{2}\gamma^2\frac{L^2}{(\gamma L-1)^2}\rho A, \tag{10a}$$

$$K_2 \geqslant \frac{1}{2}\frac{1-2\gamma L}{(\gamma L-1)^2}\gamma L\rho A, \tag{10b}$$

and $\gamma L < \frac{1}{2}$. *Then the functional* $V(t)$ *along the solution of the system, equations* (3) *and* (4a–c), *satisfies*

$$V(t) \leqslant V(0)\exp\left(-\frac{\gamma}{1+\gamma\gamma_1}\mathrm{t}\right). \tag{11}$$

Furthermore,

$$\rho(\mathbf{q},\mathbf{0}) \leqslant \sqrt{\frac{V(0)}{1-\gamma\gamma_1}}\exp\left(-\frac{\gamma}{2(1+\gamma\gamma_1)}t\right), \tag{12}$$

where $\gamma_1 = L \max(1, \rho A/P_0)$. *Hence, the transversal displacement, velocity and slope will exponentially tend to zero.*

Note that feedback gain from slope, K_1, might possess bounded negative values. To prove the theorem it is first necessary to establish two lemmas.

Lemma 1. *Let* γ *in equation* (7) *satisfy*

$$\gamma \leqslant \frac{1}{\gamma_1}. \tag{13}$$

Then, the functional $V(t)$ *satisfies*

$$(1 - \gamma\gamma_1)\rho^2(\mathbf{q}, \mathbf{0}) \leqslant V(t) \leqslant (1 + \gamma\gamma_1)\rho^2(\mathbf{q}, \mathbf{0}) \tag{14}$$

$\forall t \geqslant 0$. *Furthermore,* $V(t)$ *is positive definite with respect to metric* $\rho(\mathbf{q}, \mathbf{0})$ *and admits an infinitesimally upper limit.*

Proof. The following inequality is valid:

$$\begin{aligned}
\rho A \int_0^L x\eta_t(x,t)\eta_x(x,t)\,\mathrm{d}x &\leqslant \rho A \int_0^L x|\eta_t(x,t)||\eta_x(x,t)|\,\mathrm{d}x \\
&\leqslant L\left(\frac{\rho A}{2}\int_0^L \eta_t^2(x,t)\,\mathrm{d}x + \frac{\rho A}{P_0}\frac{P_0}{2}\int_0^L \eta_x^2(x,t)\,\mathrm{d}x\right) \\
&\leqslant L\max\left(1, \frac{\rho A}{P_0}\right)\left(\frac{\rho A}{2}\int_0^L \eta_t^2(x,t)\,\mathrm{d}x + \frac{P_0}{2}\int_0^L \eta_x^2(x,t)\,\mathrm{d}x\right) \\
&\leqslant \gamma_1 \rho^2(\mathbf{q}, \mathbf{0})
\end{aligned}$$

$\forall t \geqslant 0$, where the second inequality is obtained using the following inequality:

$$|\eta_t(x,t)||\eta_x(x,t)| \leqslant \tfrac{1}{2}(\eta_t^2(x,t) + \eta_x^2(x,t)).$$

Hence, equation (7) can be written as

$$V(t) \leqslant \rho^2(\mathbf{q}, \mathbf{0}) + \gamma\gamma_1\rho^2(\mathbf{q}, \mathbf{0}) = (1 + \gamma\gamma_1)\rho^2(\mathbf{q}, \mathbf{0}).$$

In a similar way the left-hand side of inequality (14) can be proved.

It is clear from the definition of the metric $\rho(\mathbf{q}, \mathbf{0})$ (equation (8)) that $\rho(\mathbf{q}, \mathbf{0})$ is positive definite; hence, from inequality (14) and with γ satisfying the inequality (13), it is concluded that the functional $V(t)$ is also positive definite. The right-hand side of inequality (14) indicates that the functional $V(t)$ has an upper limit which is given by $(1 + \gamma\gamma_1)\rho^2(\mathbf{q}, \mathbf{0})$. □

The derivative of equation (7) with respect to time is given by (for the sake of simplicity the argument (x, t) is omitted)

$$\begin{aligned}
\dot{V}(t) = \rho A \int_0^L \eta_{tt}\eta_t\,\mathrm{d}x + P_0 \int_0^L \eta_x\eta_{xt}\,\mathrm{d}x + \frac{EA}{2}\int_0^L \eta_x^3\eta_{xt}\,\mathrm{d}x \\
+ EI\int_0^L \eta_{xx}\eta_{xxt}\,\mathrm{d}x + \gamma\rho A\int_0^L [x\eta_{tt}\eta_x + x\eta_t\eta_{xt}]\,\mathrm{d}x.
\end{aligned} \tag{15}$$

Substitution of $\rho A\eta_{tt}$ from equation (3) into equation (15) yields

$$\dot{V}(t) = -\int_0^L EI\eta_{xxxx}\eta_t\,\mathrm{d}x + P_0\int_0^L \eta_{xx}\eta_t\,\mathrm{d}x + \frac{3EA}{2}\int_0^L \eta_x^2\eta_{xx}\eta_t\,\mathrm{d}x$$

$$+ P_0\int_0^L \eta_x\eta_{xt}\,\mathrm{d}x + \frac{EA}{2}\int_0^L \eta_x^3\eta_{xt}\,\mathrm{d}x + EI\int_0^L \eta_{xx}\eta_{xxt}\,\mathrm{d}x - \gamma\int_0^L EIx\eta_{xxxx}\eta_x\,\mathrm{d}x$$

$$+ \gamma P_0\int_0^L x\eta_{xx}\eta_x\,\mathrm{d}x + \gamma\frac{3EA}{2}\int_0^L x\eta_x^3\eta_{xx}\,\mathrm{d}x + \rho A\int_0^L x\eta_t\eta_{xt}\,\mathrm{d}x.$$

Before embarking on the further analysis some identities are proven.

Lemma 2. Let $\eta(x, t)$ satisfy the boundary-value problem, (equations (3) and (4a–c)) then

$$-EI\int_0^L [\eta_{xxxx}\eta_t - \eta_{xx}\eta_{xxt}]\,\mathrm{d}x$$

$$= -EI[\eta_{xxx}\eta_t|_0^L - \eta_{xx}\eta_{tx}|_0^L], \tag{16a}$$

$$\frac{EA}{2}\int_0^L [\eta_x^3\eta_{xt} + 3\eta_x^2\eta_{xx}\eta_t]\,\mathrm{d}x = \frac{EA}{2}\,\eta_x^3\eta_t|_0^L, \tag{16b}$$

$$P_0\int_0^L [\eta_{xx}\eta_t + \eta_x\eta_{xt}]\,\mathrm{d}x = P_0\eta_x\eta_t|_0^L, \tag{16c}$$

$$-\gamma EI\int_0^L x\eta_{xxxx}\eta_x\,\mathrm{d}x = -\gamma\frac{3EI}{2}\int_0^L \eta_{xx}^2\,\mathrm{d}x$$

$$-\gamma EI\left[L\left(\eta_{xxx}(L,t)\eta_x(L,t) - \tfrac{1}{2}\eta_{xx}^2(L,t)\right) - \eta_{xx}\eta_x|_0^L\right], \tag{16d}$$

$$\gamma P_0\int_0^L x\eta_{xx}\eta_x\,\mathrm{d}x = \gamma P_0\left[\frac{L}{2}\eta_x^2(L,t) - \tfrac{1}{2}\int_0^L \eta_x^2\,\mathrm{d}x\right], \tag{16e}$$

$$\gamma\tfrac{3}{2}EA\int_0^L x\eta_x^3\eta_{xx}\,\mathrm{d}x = \gamma\,\tfrac{3}{2}EA\left(\frac{L}{4}\eta_x^4(L,t) - \tfrac{1}{4}\int_0^L \eta_x^4\,\mathrm{d}x\right), \tag{16f}$$

$$\gamma\rho A\int_0^L x\eta_t\eta_{tx}\,\mathrm{d}x = \gamma\rho A\left[\frac{L}{2}\eta_t^2(L,t) - \tfrac{1}{2}\int_0^L \eta_t^2\,\mathrm{d}x\right], \quad \forall t \geqslant 0. \tag{16g}$$

Proof. The proof is straightforward by applying integration by parts and will not be given here. □

Now using the results of Lemma 2 and collecting all terms, an expression for $\dot{V}(t)$ *is*

$$\begin{aligned}\dot{V}(t) = &-\frac{\gamma}{2}\rho A\int_0^L \eta_t^2\,\mathrm{d}x - \gamma\frac{P_0}{2}\int_0^L \eta_x^2\,\mathrm{d}x\\ &-\gamma\frac{3EA}{8}\int_0^L \eta_x^4\,\mathrm{d}x - \gamma\frac{3EI}{2}\int_0^L \eta_{xx}^2\,\mathrm{d}x\\ &- EI[\eta_{xxx}(L,t)\eta_t(L,t) - \eta_{xxx}(0,t)\eta_t(0,t)]\\ &+ EI[\eta_{xx}(L,t)\eta_{tx}(L,t) - \eta_{xx}(0,t)\eta_{tx}(0,t)]\\ &+\frac{EA}{2}[\eta_x^3(L,t)\eta_t(L,t) - \eta_x^3(0,t)\eta_t(0,t)]\\ &+ P_0[\eta_x(L,t)\eta_t(L,t) - \eta_x(0,t)\eta_t(0,t)]\\ &+\gamma L\frac{3EA}{8}\eta_x^4(L,t) + \gamma L\frac{P_0}{2}\eta_x^2(L,t) + \gamma\frac{L}{2}\rho A\eta_t^2(L,t)\\ &-\gamma LEI\left(\eta_{xxx}(L,t)\eta_x(L,t) - \tfrac{1}{2}\eta_{xx}^2(L,t)\right)\\ &+\gamma EI[\eta_{xx}(L,t)\eta_x(L,t) - \eta_{xx}(0,t)\eta_x(0,t)]. \end{aligned} \quad (17)$$

From the boundary condition (4b), $\eta_t(0,t) = 0\,\forall t \geqslant 0$. Using, boundary condition (4a) and substituting for $EI\eta_{xxx}(L,t)$ from equation (4c) into equation (17) results in

$$\begin{aligned}\dot{V}(t) = &-\frac{\gamma}{2}\rho A\int_0^L \eta_t^2\,\mathrm{d}x - \gamma\frac{P_0}{2}\int_0^L \eta_x^2\,\mathrm{d}x\\ &-\gamma\frac{3}{8}EA\int_0^L \eta_x^4\,\mathrm{d}x - \gamma\frac{3}{2}EI\int_0^L \eta_{xx}^2\,\mathrm{d}x\\ &-\gamma L\frac{EA}{8}\eta_x^4(L,t) - \gamma\frac{\mathrm{L}}{2}P_0\eta_x^2(L,t)\\ &+\gamma Lu(t)\eta_x(L,t) + \gamma\frac{L}{2}\rho A\eta_t^2(L,t)\\ &+ u(t)\eta_t(L,t). \end{aligned} \quad (18)$$

Theorem 2 may now be proved. □

Proof of Theorem 2. Substitution of the control law equation (9) into equation (18) and collecting terms yield

$$\dot{V}(t) = -\frac{\gamma}{2}\rho A \int_0^L \eta_t^2\,\mathrm{d}x - \gamma\frac{P_0}{2}\int_0^L \eta_x^2\,\mathrm{d}x$$

$$-\gamma\frac{3}{8}EA\int_0^L \eta_x^4\,\mathrm{d}x - \gamma\tfrac{3}{2}EI\int_0^L \eta_{xx}^2\,\mathrm{d}x$$

$$-\gamma L\frac{EA}{8}\eta_x^4(L,t) - \gamma\frac{\mathrm{L}}{2}P_0\eta_x^2(L,t)$$

$$-\gamma LK_1\eta_x^2(L,t) - \left(\mathrm{K}_2 - \gamma\frac{\mathrm{L}}{2}\rho\mathrm{A}\right)\eta_t^2(L,t)$$

$$-(K_1+\gamma LK_2)\eta_x(L,t)\eta_t(L,t).$$

Using the following inequality,

$$\eta_x(L,t)\eta_t(L,t) \geqslant \tfrac{1}{2}\eta_x^2(L,t) - \tfrac{1}{2}\eta_t^2(L,t),$$

$\dot{V}(t)$ can be rewritten as

$$\dot{V}(t) \leqslant -\frac{\gamma}{2}\rho A \int_0^L \eta_t^2\,\mathrm{d}x - \gamma\frac{P_0}{2}\int_0^L \eta_x^2\,\mathrm{d}x$$

$$-\gamma\frac{3}{8}EA\int_0^L \eta_x^4\,\mathrm{d}x - \gamma\frac{3}{2}EI\int_0^L \eta_{xx}^2\,\mathrm{d}x$$

$$-\gamma L\frac{EA}{8}\eta_x^4(L,t) - \gamma\frac{\mathrm{L}}{2}P_0\eta_x^2(L,t)$$

$$-\left(\gamma LK_1 - \frac{K_1+\gamma LK_2}{2}\right)\eta_x^2(L,t)$$

$$-\left(K_2 - \gamma\frac{L}{2}\rho A - \frac{K_1+\gamma LK_2}{2}\right)\eta_t^2(L,t).$$

Selecting control gains K_1 and K_2 according to equations (10a) and (10b) renders $\dot{V}(t)$ negative definite and hence,

$$\dot{V}(t) \leqslant -\gamma\rho^2(\mathbf{q},\mathbf{0}).$$

Now, using inequality (14) gives

$$\dot{V}(t) \leqslant -\frac{\gamma}{1+\gamma\gamma_1}V(t).$$

By the comparison lemma [13], inequality (11) the theorem is proven. From inequality (11), using inequalities (13) and (14), it is easy to obtain inequality (12). Consequently, the slope $\eta_x(x,t)$ and the velocity $\eta_t(x,t)$ will exponentially go toward zero at $t \to 0$ for all $x \in [0, L]$.

Thus, the deflection $\eta(x, t)$ will also converge exponentially to zero as $t \to 0$ for all $x \in [0, L]$ since $\eta(0, t)$ for all $t \geqslant 0$. $\square$

4.1. IMPLEMENTATIONAL ASPECT OF THE CONTROLLER

The boundary feedback controller equation (9) consists of feedback from the slope and velocity on the right-hand side of the beam. An implementation of this controller should be made non-dimensional with dimensional upscaling of the control signal. This will improve the numerics and make the control performance comparable with other controllers.

5. CONCLUSION

A boundary control law is designed to stabilize the transversal vibration of a beam exponentially. Exponential stability is proved by using a Lyapunov functional. It has been shown that the mechanical energy of the system will go exponentially toward zero. Since the control law consists only of feedback from the slope and velocity of the beam at the boundary, measurement cost is minimized and deterioration effect of spillover phenomena are avoided.

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